

# HOMOLOGICAL PROJECTIVE DUALITY FOR LINEAR SYSTEMS WITH BASE LOCUS

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**ABSTRACT.** We show how blowing up varieties in base loci of linear systems gives a procedure for creating new homological projective duals from old.

Starting with a HP dual pair  $X, Y$  and smooth orthogonal linear sections  $X_L, Y_L$ , we prove that the blowup of  $X$  in  $X_L$  is naturally HP dual to  $Y_L$ . The result does not need  $Y$  to exist as a variety, i.e. it may be "non-commutative".

We extend the result to the case where the base locus  $X_L$  is a multiple of a smooth variety and the universal hyperplane has rational singularities; here the HP dual is a categorical resolution of singularities of  $Y_L$ .

Finally we give examples where, starting with a noncommutative  $Y$ , the above process nevertheless gives geometric HP duals.

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## 1. INTRODUCTION

One of the most powerful tools for investigating derived categories of algebraic varieties and their semiorthogonal decompositions is Kuznetsov's homological projective duality [Kuz07]. It is a beautiful theory but it's hard to produce geometric examples; however some actual examples do exist [Kuz08a, Kuz06a, Kuz10, BBF14, Ren15]. HP duality starts with a base point free linear system on some variety  $X$ ; based on a guess of Calabrese and Thomas [CT15] we see that for a sublinear system with base locus one can consider a natural HPD problem on the blowup of  $X$  in the base locus. We show that this new HPD problem is closely related to the original one. In particular we obtain a procedure for constructing new HP duals from old:

**Theorem 1.1.** *Let  $X \rightarrow \mathbb{P}(V)$  and  $Y \rightarrow \mathbb{P}(V^*)$  be a HP dual pair with respect to the Lefschetz decomposition of  $\mathbf{D}^b(X)$  given by*

$$\mathbf{D}^b(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{i-1}(i-1) \rangle.$$

*Then for any sublinear system  $L \subset V^*$ , such that the corresponding linear sections  $X_L, Y_L$  have expected dimension, we have an HP dual pair  $\mathrm{Bl}_{X_L} X \rightarrow \mathbb{P}(L^*)$  and  $Y_L \rightarrow \mathbb{P}(L)$  with respect to the following Lefschetz decomposition of  $\mathbf{D}^b(\mathrm{Bl}_{X_L} X)$ :*

$$\mathbf{D}^b(\mathrm{Bl}_{X_L} X) = \langle \tilde{\mathcal{A}}_0, \tilde{\mathcal{A}}_1(1), \dots, \tilde{\mathcal{A}}_{l-2}(l-2) \rangle,$$

where  $l = \dim L$  and the pieces  $\tilde{\mathcal{A}}_k$  are

$$\tilde{\mathcal{A}}_k := \begin{cases} \langle \pi^* \mathcal{A}_k \otimes \mathcal{O}((l-1)E), \mathbf{D}^b(X_L)_{-l+1} \rangle & \text{if } k < i, \\ \mathbf{D}^b(X_L)_{-l+1} & \text{if } k \geq i. \end{cases}$$

Here  $\pi: \mathrm{Bl}_{X_L} X \rightarrow X$  is the projection from the blowup and  $E$  is the exceptional divisor.

The fact that this is true can be seen almost immediately by thinking fibrewise. Indeed, by assumption we know that the interesting part of the derived category of a hyperplane section  $X_H$  of  $X \rightarrow \mathbb{P}(V)$  is just the derived category of the corresponding fibre of  $Y \rightarrow \mathbb{P}(V^*)$ :

$$\mathbf{D}^b(X_H) = \langle \mathbf{D}^b(Y_H), \mathcal{A}(1), \dots, \mathcal{A}(i-1) \rangle.$$

Obviously a hyperplane section of  $\mathrm{Bl}_{X_L} X \rightarrow \mathbb{P}(L^*)$  is just  $\mathrm{Bl}_{X_L} X_H$  and thus by Orlov's theorem [Orl93] one notices that the interesting part of its derived category is also  $\mathbf{D}^b(Y_H)$ . However, we are restricting our attention to only those hyperplane sections that contain  $X_L$  and thus the expected HP dual must be the restriction  $Y_L \rightarrow \mathbb{P}(L)$  of the original one.

The proof now consists of making the above work in families by writing the universal hyperplane section  $\tilde{\mathcal{H}}$  of the blowup as a blowup itself:

$$(1) \quad \tilde{\mathcal{H}} = \mathrm{Bl}_{X_L \times \mathbb{P}(L)} \mathcal{H}_L.$$

Afterwards it's just a matter of applying Orlov's theorem and performing some mutations to obtain the result. The whole story works actually more generally if we start with a noncommutative HP dual. Furthermore, if we start with a linear system  $L$  whose base locus is of the form  $mZ$  for some smooth variety  $Z$  with  $m \geq 1$ , then we have the following:

**Theorem 1.2.** *There is a natural Lefschetz decomposition for  $\mathbf{D}^b(\mathrm{Bl}_Z X)$  with respect to the line bundle  $\pi^* \mathcal{O}_X(1)(-mE)$  such that, if  $\mathcal{H}_L$  has only rational singularities, the HP dual of  $\mathrm{Bl}_Z X \rightarrow \mathbb{P}(L)$  is a categorical resolution of singularities [Kuz08b] of the interesting part of  $\mathbf{D}^b(\mathcal{H}_L)$ .*

The idea and the techniques of the proof are basically the same as in Theorem 1.1. We still have the isomorphism (1) which tells us that  $\mathbf{D}^b(\tilde{\mathcal{H}})$  is a categorical resolution of singularities of  $\mathbf{D}^b(\mathcal{H}_L)$ . Restricting to the relevant subcategories the result immediately follows.

Allowing base locus with multiplicity leads us to consider the final two examples where we see an interesting phenomenon: starting with a noncommutative HP dual the blowing up process yields a geometric HP dual pair. More precisely we consider the degree 3 Veronese embedding

$$\mathbb{P}^5 \hookrightarrow \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^5}(3))^*)$$

with the standard Lefschetz decomposition of  $\mathbf{D}^b(\mathbb{P}^5)$  with respect to  $\mathcal{O}_{\mathbb{P}^5}(3)$ . It is known that the HP dual in this case is a noncommutative K3-fibration

$$\mathcal{C} \rightarrow \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^5}(3))).$$

By setting  $L$  to be the linear system of cubics that are singular at a fixed point  $P \in \mathbb{P}^5$  we obtain a base point free linear system  $L = H^0(\pi^* \mathcal{O}_{\mathbb{P}^5}(3)(-2E))$  on  $\mathrm{Bl}_P \mathbb{P}^5$ , where  $\pi: \mathrm{Bl}_P \mathbb{P}^5 \rightarrow \mathbb{P}^5$  is the blowup and  $E$  is its exceptional divisor. We can equip  $\mathbf{D}^b(\mathrm{Bl}_P \mathbb{P}^5)$  with a Lefschetz decomposition (similar to the one in Theorem 1.1) with respect to the line bundle  $\pi^* \mathcal{O}_{\mathbb{P}^5}(3)(-2E)$ . We show that the HP dual of  $\mathrm{Bl}_P \mathbb{P}^5 \rightarrow \mathbb{P}(L^*)$  is generically a K3-fibration  $\check{X} \rightarrow \mathbb{P}(L)$ , where  $\check{X} \subset \mathbb{P}^4 \times \mathbb{P}(L)$  is the intersection of a universal  $(2, 1)$  and a universal  $(3, 1)$  divisor.

For the second example we take  $L$  to be the linear system of cubics containing a fixed plane  $\mathbb{P}(W) \cong \mathbb{P}^2 \subset \mathbb{P}^5$ . The HP dual we obtain in this case is the noncommutative variety  $(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0)$ , where  $\mathbb{P}(W')$  is the orthogonal complement of  $\mathbb{P}(W) \subset \mathbb{P}^5$  and  $\mathcal{C}_0$  is an even Clifford algebra sheaf on  $\mathbb{P}(W') \times \mathbb{P}(L)$ . Both of these results are straightforward extensions of Kuznetsov's results on cubic fourfolds [Kuz10].

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## 2. PRELIMINARIES

**2.1. Notation and conventions.** Let  $X$  be an algebraic variety. We always work over an algebraically closed field  $\mathbb{K}$  of characteristic zero. We will denote the bounded derived category of coherent sheaves on  $X$  by  $\mathbf{D}^b(X)$ . We will abuse notation and denote the total derived tensor product by  $\otimes$  and the total derived pushforward and pullback of a map  $f$  by  $f_*$  and  $f^*$ , respectively. For a sheaf  $\mathcal{F}$  on  $X$  we will denote its dual by  $\mathcal{F}^\vee$ , whereas for a vector space  $V$  we will denote its dual by  $V^*$ . For two objects  $\mathcal{F}, \mathcal{G} \in \mathbf{D}^b(X)$  we will denote by  $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$  the set of morphisms from  $\mathcal{F}$  to  $\mathcal{G}$  in  $\mathbf{D}^b(X)$ , whereas  $\mathbf{R}\mathrm{Hom}(\mathcal{F}, \mathcal{G})$  shall denote the derived global Hom. In particular we identify

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}[k]) = H^0(\mathbf{R}\mathrm{Hom}(\mathcal{F}, \mathcal{G}[k])) = \mathrm{Ext}^k(\mathcal{F}, \mathcal{G}).$$

**2.2. Homological projective duality.** Consider a smooth projective variety  $X$  together with a regular map  $f: X \rightarrow \mathbb{P}(V)$  for some finite-dimensional vector space  $V$ . Without loss of generality we will assume that the image of  $f$  is not contained in a hyperplane. Note that this is equivalent to giving an effective line bundle  $\mathcal{O}_X(1) := f^*\mathcal{O}_{\mathbb{P}(V)}(1)$  on  $X$  together with a base point free linear system  $V^* \subseteq H^0(\mathcal{O}_X(1))$ .

**Definition 2.1.** A *Lefschetz decomposition* of  $\mathbf{D}^b(X)$  with respect to a fixed line bundle  $\mathcal{O}_X(1)$  is a semiorthogonal decomposition of the form

$$(2) \quad \mathbf{D}^b(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{i-1}(i-1) \rangle,$$

where  $0 \subset \mathcal{A}_{i-1} \subset \dots \subset \mathcal{A}_0 \subset \mathbf{D}^b(X)$  are full triangulated subcategories. Here,

$$\mathcal{A}_k(k) = \{ \mathcal{F} \otimes \mathcal{O}_X(k) \mid \mathcal{F} \in \mathcal{A}_k \}.$$

If  $\mathcal{A}_0 = \dots = \mathcal{A}_{i-1}$  we call the Lefschetz decomposition *rectangular*.

*Remark 2.2.* Recall that a Lefschetz decomposition gets its name from the fact that for any hyperplane section  $X_H$  of  $X \rightarrow \mathbb{P}(V)$  we have:

- (1) The restriction of the derived pullback functor  $i^*|_{\mathcal{A}_k(k)}: \mathcal{A}_k(k) \rightarrow \mathbf{D}^b(X_H)$  is fully faithful for  $k \geq 1$ ,
- (2)  $i^*(\mathcal{A}_1(1)), \dots, i^*(\mathcal{A}_{i-1}(i-1))$  form a semiorthogonal collection in  $\mathbf{D}^b(X_H)$ .

Consequently we have a semiorthogonal decomposition of the form

$$\mathbf{D}^b(X_H) = \langle \mathcal{C}_H, \mathcal{A}_1(1), \dots, \mathcal{A}_{i-1}(i-1) \rangle,$$

for some triangulated subcategory  $\mathcal{C}_H$ .

Let  $\mathcal{H}$  be the universal hyperplane section of  $X \rightarrow \mathbb{P}(V)$ . Explicitly this means

$$\mathcal{H} = \{ (p, [s]) \in X \times \mathbb{P}(V^*) \mid s(p) = 0 \},$$

which is the zero locus of the tautological section in

$$H^0(\mathcal{O}_{X \times \mathbb{P}(V^*)}(1, 1)) \cong H^0(\mathcal{O}_X(1)) \otimes V.$$

We now recall Kuznetsov's definition of *homological projective duals*:

**Definition 2.3.** Let  $X \rightarrow \mathbb{P}(V)$  be equipped with a Lefschetz decomposition as above. A projective variety  $Y$  together with a regular map  $Y \rightarrow \mathbb{P}(V^*)$  is called *homological projective dual* of  $X$  if there is an object  $\mathcal{F} \in \mathbf{D}^b(Y \times_{\mathbb{P}(V^*)} \mathcal{H})$  such that the corresponding Fourier–Mukai transform  $\Phi_{Y \rightarrow \mathcal{H}}^{\mathcal{F}}: \mathbf{D}^b(Y) \rightarrow \mathbf{D}^b(\mathcal{H})$  is fully faithful and there is a semiorthogonal decomposition

$$\mathbf{D}^b(\mathcal{H}) = \langle \Phi_{Y \rightarrow \mathcal{H}}^{\mathcal{F}}(\mathbf{D}^b(Y)), \mathcal{A}_1(1) \boxtimes \mathbf{D}^b(\mathbb{P}(V^*)), \dots, \mathcal{A}_{i-1}(i-1) \boxtimes \mathbf{D}^b(\mathbb{P}(V^*)) \rangle.$$

Here  $\mathcal{A}_k(k) \boxtimes \mathbf{D}^b(\mathbb{P}(V^*))$  is shorthand for  $i_{\mathcal{H}}^*(\mathcal{A}_k(k) \boxtimes \mathbf{D}^b(\mathbb{P}(V^*)))$ , where  $i_{\mathcal{H}}$  is the inclusion of the divisor  $\mathcal{H} \subset X \times \mathbb{P}(V^*)$ .

Let  $L \subset V^*$  be a sublinear system and let  $l$  denote its dimension. In the following we denote its base locus by  $X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp)$ . For a homological projective dual  $Y \rightarrow \mathbb{P}(V^*)$  we will write  $Y_L := Y \times_{\mathbb{P}(V^*)} \mathbb{P}(L)$ . Kuznetsov's main theorem on homological projective duality [Kuz07] now tells us precisely what Definition 2.3 does for us in terms of the linear sections  $X_L$  of  $X$ :

**Theorem 2.4** (Kuznetsov). *Let  $X \rightarrow \mathbb{P}(V)$  and its Lefschetz decomposition be as above and assume  $Y \rightarrow \mathbb{P}(V^*)$  is a homologically projectively dual variety. Assume furthermore that  $\dim V > i$ , where  $i$  is the number of terms in the Lefschetz decomposition of  $\mathbf{D}^b(X)$ . Then we have the following facts:*

- (1)  $Y$  is smooth and  $\mathbf{D}^b(Y)$  has a dual Lefschetz decomposition given by

$$\mathbf{D}^b(Y) = \langle \mathcal{B}_{j-1}(1-j), \dots, \mathcal{B}_1(-1), \mathcal{B}_0 \rangle,$$

with  $0 \subset \mathcal{B}_{j-1} \subset \dots \subset \mathcal{B}_0 \subset \mathbf{D}^b(Y)$  and  $j = \dim V - \max\{k \mid \mathcal{A}_k = \mathcal{A}_0\}$ . Here the  $\mathcal{B}_k$  are twisted by the pullback of  $\mathcal{O}_{\mathbb{P}(V^*)}(1)$ .

- (2) If  $\dim X_L = \dim X - l$  and  $\dim Y_L = \dim Y - \dim V + l$  then there is a triangulated category  $\mathcal{C}_L$  such that we have semiorthogonal decompositions

$$\mathbf{D}^b(X_L) = \langle \mathcal{C}_L, \mathcal{A}_l(1), \dots, \mathcal{A}_{i-1}(i-l) \rangle$$

$$\mathbf{D}^b(Y_L) = \langle \mathcal{B}_{j-1}(\dim V - l - j), \dots, \mathcal{B}_{\dim V - l}(-1), \mathcal{C}_L \rangle.$$

*Remark 2.5.* Note that a *categorical*<sup>1</sup> HP dual  $\mathcal{C}$  always exists tautologically. Namely we can just define it to be the right orthogonal of the trivial part of  $\mathbf{D}^b(\mathcal{H})$ :

$$\mathcal{C} := \langle \mathcal{A}_1(1) \boxtimes \mathbf{D}^b(\mathbb{P}(V^*)), \dots, \mathcal{A}_{i-1}(i-1) \boxtimes \mathbf{D}^b(\mathbb{P}(V^*)) \rangle^\perp.$$

One can then still develop much of the general theory in this categorical setting; we refer to Thomas' notes [Tho] for a good introduction to that point of view.

### 3. LINEAR SYSTEMS WITH BASE LOCUS

Assume we are given  $f: X \rightarrow \mathbb{P}(V)$  with some Lefschetz decomposition and a corresponding homological projective dual  $Y \rightarrow \mathbb{P}(V^*)$ . Also assume we have a linear subspace  $L \subset V^*$  of dimension  $l$  such that  $X_L$  is smooth,  $\dim X_L = \dim X - l$ , and  $\dim Y_L = \dim Y - \dim V + l$ . Note that there is a natural map  $Y_L \rightarrow \mathbb{P}(L)$  and

<sup>1</sup>We will often use "categorical" and "noncommutative" interchangeably.

a rational map  $X \dashrightarrow \mathbb{P}(L^*)$ . We consider the blow up of the rational map in its indeterminacy locus  $X_L$  to obtain a regular map:

$$(3) \quad \begin{array}{ccc} E & \xhookrightarrow{j} & \mathrm{Bl}_{X_L} X \\ \downarrow p & & \downarrow \pi \\ X_L & \xhookrightarrow{i} & X \end{array} \quad \begin{array}{c} \searrow \phi \\ \dashrightarrow \mathbb{P}(L^*) \end{array}$$

where by construction we have

$$\phi^* \mathcal{O}_{\mathbb{P}(L^*)}(1) \cong \pi^* \mathcal{O}_X(1)(-E),$$

and  $E$  is the exceptional divisor of the blowup. From now on write  $\tilde{X} := \mathrm{Bl}_{X_L} X$  and we obviously always assume  $l \geq 2$ . In order to show that  $\tilde{X} \rightarrow \mathbb{P}(L^*)$  and  $Y_L \rightarrow \mathbb{P}(L)$  are HP dual we have to fix a Lefschetz decomposition for  $\mathbf{D}^b(\tilde{X})$ . For this we will use Orlov's decomposition for a blowup [Orl93].

**3.1. Stupid Lefschetz decomposition.** We first look at the easiest case: when  $\mathbf{D}^b(X)$  is endowed with the stupid Lefschetz decomposition and the homological projective dual is just the universal hyperplane section  $\mathcal{H} \rightarrow \mathbb{P}(V^*)$ . In this case Orlov's theorem then gives us a semiorthogonal decomposition of the form

$$\mathbf{D}^b(\tilde{X}) = \langle \pi^* \mathbf{D}^b(X) \otimes \omega_{\tilde{X}}, \mathbf{D}^b(X_L)_{-l+1}, \dots, \mathbf{D}^b(X_L)_{-1} \rangle.$$

If we now set

$$(4) \quad \tilde{\mathcal{A}}_0 := \langle \pi^* \mathbf{D}^b(X) \otimes \omega_{\tilde{X}}, \mathbf{D}^b(X_L)_{-l+1} \rangle, \quad \tilde{\mathcal{A}}_1 := \dots := \tilde{\mathcal{A}}_{l-2} := \mathbf{D}^b(X_L)_{-l+1},$$

then we have a Lefschetz decomposition of  $\mathbf{D}^b(\tilde{X})$ :

**Proposition 3.1.** *The above is a Lefschetz decomposition of  $\mathbf{D}^b(\tilde{X})$  with respect to the line bundle  $\pi^* \mathcal{O}_X(1)(-E)$ , i.e. we have a semiorthogonal decomposition of the form*

$$\mathbf{D}^b(\tilde{X}) = \langle \tilde{\mathcal{A}}_0, \tilde{\mathcal{A}}_1(1), \dots, \tilde{\mathcal{A}}_{l-2}(l-2) \rangle,$$

with  $0 \subset \tilde{\mathcal{A}}_{l-2} \subset \dots \subset \tilde{\mathcal{A}}_0 \subset \mathbf{D}^b(\tilde{X})$ .

*Proof.* All that's left to show is that  $\mathbf{D}^b(X_L)_k \otimes \pi^* \mathcal{O}_X(1)(-E) = \mathbf{D}^b(X_L)_{k+1}$ , for any  $k$ . But this follows immediately from  $j^* \mathcal{O}(-E) \cong \mathcal{O}_E(1)$  and the projection formula.  $\square$

**Proposition 3.2.** *Let  $\tilde{X} \rightarrow \mathbb{P}(L^*)$  be endowed with the Lefschetz decomposition (4). Then its homological projective dual is  $\mathcal{H}_L \rightarrow \mathbb{P}(L)$ .*

*Proof.* Let  $\tilde{\mathcal{H}}$  denote the universal hyperplane section of  $\tilde{X}$  with respect to the line bundle  $\pi^* \mathcal{O}_X(1)(-E)$ . We need to show that there is an object  $\mathcal{F} \in \mathbf{D}^b(\mathcal{H}_L \times_{\mathbb{P}(L)} \tilde{\mathcal{H}})$  such that its associated Fourier–Mukai functor  $\Phi_{\mathcal{H}_L \rightarrow \tilde{\mathcal{H}}}^{\mathcal{F}} : \mathbf{D}^b(\mathcal{H}_L) \rightarrow \mathbf{D}^b(\tilde{\mathcal{H}})$  is fully faithful and we have a semiorthogonal decomposition

$$\mathbf{D}^b(\tilde{\mathcal{H}}) = \langle \Phi_{\mathcal{H}_L \rightarrow \tilde{\mathcal{H}}}^{\mathcal{F}}(\mathbf{D}^b(\mathcal{H}_L)), \mathbf{D}^b(X_L)_{-l+2} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots, \mathbf{D}^b(X_L)_{-1} \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle.$$

The easiest way to do that is by observing that  $\tilde{\mathcal{H}} \cong \mathrm{Bl}_{X_L \times \mathbb{P}(L)} \mathcal{H}_L$ . Thinking fibrewise this is obvious: hyperplane sections of  $\phi$  are just strict transforms of hyperplane sections of  $f: X \rightarrow \mathbb{P}(V)$  containing the base locus  $X_L$ . In families, just look at  $\mathcal{H}_L \in X \times \mathbb{P}(V)$ , and repeat the argument to say that  $\tilde{\mathcal{H}}$  is the strict

transform in  $\pi \times \text{id}_{\mathbb{P}(L)}: \text{Bl}_{X_L} X \times \mathbb{P}(L) \rightarrow X \times \mathbb{P}(L)$  of  $\mathcal{H}_L$ . We will denote the projection by  $\tilde{\pi}: \tilde{\mathcal{H}} \rightarrow \mathcal{H}_L$  and we have the commutative diagram

$$(5) \quad \begin{array}{ccccc} & \tilde{E} & \xrightarrow{\tilde{j}} & \tilde{\mathcal{H}} & \\ & \swarrow & & \nwarrow i_{\tilde{\mathcal{H}}} & \\ E \times \mathbb{P}(L) & \xrightarrow{j \times \text{id}_{\mathbb{P}(L)}} & \tilde{X} \times \mathbb{P}(L) & & \\ \downarrow p \times \text{id}_{\mathbb{P}(L)} & & \downarrow \tilde{p} & & \downarrow \tilde{\pi} \\ & X_L \times \mathbb{P}(L) & \xrightarrow{\pi \times \text{id}_{\mathbb{P}(L)}} & \mathcal{H}_L & \\ \downarrow p \times \text{id}_{\mathbb{P}(L)} & & \downarrow \pi \times \text{id}_{\mathbb{P}(L)} & & \downarrow i_{\mathcal{H}_L} \\ X_L \times \mathbb{P}(L) & \xrightarrow{i \times \text{id}_{\mathbb{P}(L)}} & X \times \mathbb{P}(L) & & \end{array},$$

where the front and back faces are blowup diagrams. Since the codimension of  $X_L \times \mathbb{P}(L)$  in  $\mathcal{H}_L$  is  $l - 1$ , applying Orlov's theorem to  $\tilde{\mathcal{H}}$  we obtain the semiorthogonal decomposition

$$\mathbf{D}^b(\tilde{\mathcal{H}}) = \langle \tilde{\pi}^* \mathbf{D}^b(\mathcal{H}_L) \otimes \omega_{\tilde{\mathcal{H}}}, \mathbf{D}^b(X_L \times \mathbb{P}(L))_{-l+2}, \dots, \mathbf{D}^b(X_L \times \mathbb{P}(L))_{-1} \rangle.$$

Note that we have  $\mathcal{O}(\tilde{E}) \cong i_{\tilde{\mathcal{H}}}^*(\mathcal{O}(E) \boxtimes \mathcal{O}_{\mathbb{P}(L)})$  and the top face of the diagram (5) is an exact Cartesian square, and thus one can easily compute

$$\mathbf{D}^b(X_L \times \mathbb{P}(L))_k = i_{\tilde{\mathcal{H}}}^*(\mathbf{D}^b(X_L)_k \boxtimes \mathbf{D}^b(\mathbb{P}(L))).$$

Finally note that  $\tilde{\pi}(-) \otimes \omega_{\tilde{\mathcal{H}}}$  can obviously be written as a Fourier–Mukai functor with kernel pushed forward from the fiber product  $\tilde{\mathcal{H}} \times_{\mathbb{P}(L)} \mathcal{H}_L$ . Indeed the kernel is just given by the pushforward of the canonical bundle along the graph map:

$$(\tilde{\pi} \times \text{id}_{\mathcal{H}_L})_* \omega_{\tilde{\mathcal{H}}}.$$

As the graph map is linear over  $\mathbb{P}(L)$  the pushforward factors via  $\tilde{\mathcal{H}} \times_{\mathbb{P}(L)} \mathcal{H}_L$ .  $\square$

*Remark 3.3.* Note that to use Orlov's decomposition in the proof we implicitly used the fact that  $\mathcal{H}_L$  is smooth whenever  $X_L$  is. Indeed, notice that  $\mathcal{H}_L \rightarrow X$  is a smooth  $\mathbb{P}^{l-2}$ -bundle away from  $X_L$ , and  $\mathcal{H}_L \rightarrow \mathbb{P}(L)$  is a smooth bundle near  $X_L$ : every element of the linear system  $L$  is smooth near the base locus  $X_L$ .

**3.2. General Lefschetz decomposition.** Assume now that  $\mathbf{D}^b(X)$  is endowed with an arbitrary Lefschetz decomposition of the form

$$(6) \quad \mathbf{D}^b(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{i-1}(i-1) \rangle,$$

and that  $Y \rightarrow \mathbb{P}(V^*)$  is a corresponding homological projective dual. Again we are hoping that  $\tilde{X} \rightarrow \mathbb{P}(L^*)$  equipped with some Lefschetz decomposition is homologically projectively dual to  $Y_L \rightarrow \mathbb{P}(L)$ . If we want to construct a Lefschetz decomposition of  $\mathbf{D}^b(\tilde{X})$  using all the information from (6), one reasonable way of doing that is by interlacing the pieces from (6) and the pieces  $\mathbf{D}^b(X_L)_k$  coming from Orlov's theorem. Note that this way we expect to get a Lefschetz decomposition with  $\max\{i, l-1\}$  terms. Thus to make Theorem 2.4 apply we will assume from now on  $i \leq l-1$ .

In order to interlace the pieces we will need to perform a series of mutations. The necessary mutations can be described generally in any semiorthogonal decomposition coming from Orlov's theorem:

**Proposition 3.4.** *Consider the blowup of a smooth variety  $X$  in a smooth subvariety  $Z$  and let the notation be analogous to the one in the diagram (3). Then for any  $\mathcal{F} \in \mathbf{D}^b(X)$  and any integer  $k$  we have the following isomorphism:*

$$\mathbb{L}_{\mathbf{D}^b(Z)_{-k}}(\pi^* \mathcal{F} \otimes \mathcal{O}((k-1)E)) \cong \pi^* \mathcal{F} \otimes \mathcal{O}(kE).$$

*Proof.* We will write  $\Phi_k$  for the embedding of  $\mathbf{D}^b(Z)$  into  $\mathbf{D}^b(\mathrm{Bl}_Z X)$ , i.e.

$$\Phi_k(-) := j_* p^*(-) \otimes \mathcal{O}(kE).$$

Now recall that left mutation through  $\Phi_k(\mathbf{D}^b(Z))$  is by definition given by the distinguished triangle

$$\Phi_k(\Phi_k^!(\mathcal{F})) \rightarrow \mathcal{F} \rightarrow \mathbb{L}_{\Phi_k(\mathbf{D}^b(Z))}(\mathcal{F}) \rightarrow \Phi_k(\Phi_k^!(\mathcal{F}))[1],$$

for any  $\mathcal{F} \in \mathbf{D}^b(\mathrm{Bl}_Z X)$ , where  $\Phi_k^!$  denotes the right adjoint of  $\Phi_k$ . Using the fact that  $j$  is an embedding of a divisor we obtain

$$\Phi_k^!(-) = p_* j^*(- \otimes \mathcal{O}((1-k)E))[-1].$$

Thus for any  $\mathcal{F} \in \mathbf{D}^b(X)$  we compute using  $p_* \mathcal{O}_E \cong \mathcal{O}_Z$ :

$$\Phi_k^!(\pi^* \mathcal{F}((k-1)E)) = p_* j^* \pi^* \mathcal{F}[-1] = p_* p^* i^* \mathcal{F}[-1] \cong i^* \mathcal{F}[-1],$$

and in particular we have

$$\Phi_k(\Phi_k^!(\pi^* \mathcal{F} \otimes \mathcal{O}((k-1)E))) \cong j_* p^* i^* \mathcal{F}(kE)[-1] \cong j_* j^* \pi^* \mathcal{F}(kE)[-1].$$

Derived tensoring the short exact sequence

$$0 \rightarrow \mathcal{O}((k-1)E) \rightarrow \mathcal{O}(kE) \rightarrow j_* j^* \mathcal{O}(kE) \rightarrow 0$$

with  $\pi^* \mathcal{F}$  and shifting the resulting distinguished triangle we obtain

$$\mathbb{L}_{\Phi_k(\mathbf{D}^b(Z))}(\pi^* \mathcal{F} \otimes \mathcal{O}((k-1)E)) \cong \pi^* \mathcal{F} \otimes \mathcal{O}(kE). \quad \square$$

Thus we obtain

**Corollary 3.5.** *If we define*

$$(7) \quad \tilde{\mathcal{A}}_k := \begin{cases} \langle \pi^* \mathcal{A}_k \otimes \mathcal{O}((l-1)E), \mathbf{D}^b(X_L)_{-l+1} \rangle & \text{if } k < i, \\ \mathbf{D}^b(X_L)_{-l+1} & \text{if } k \geq i, \end{cases}$$

for  $0 \leq k \leq l-2$ , then we get a Lefschetz decomposition of  $\mathbf{D}^b(\tilde{X})$  with respect to  $\pi^* \mathcal{O}_X(1)(-E)$ .

Note that the Lefschetz decomposition (7) indeed specialises to (4) in the case of the stupid Lefschetz decomposition on  $\mathbf{D}^b(X)$ .

**Theorem 3.6.** *Let  $\tilde{X} \rightarrow \mathbb{P}(L^*)$  be endowed with the Lefschetz decomposition (7). Then its homological projective dual is  $Y_L \rightarrow \mathbb{P}(L)$ .*

*Proof.* Unless noted otherwise we use the same notation as in Proposition 3.2. Thus we have to find an object  $\mathcal{F} \in \mathbf{D}^b(Y_L \times_{\mathbb{P}(L)} \tilde{\mathcal{H}})$  such that its associated Fourier–Mukai transform  $\Phi_{Y_L \rightarrow \tilde{\mathcal{H}}}^{\mathcal{F}}$  is fully faithful and we have a semiorthogonal decomposition

$$(8) \quad \mathbf{D}^b(\tilde{\mathcal{H}}) = \left\langle \Phi_{Y_L \rightarrow \tilde{\mathcal{H}}}^{\mathcal{F}}(\mathbf{D}^b(Y_L)), \tilde{\mathcal{A}}_1(1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots, \tilde{\mathcal{A}}_{l-2}(l-2) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \right\rangle,$$

where now the  $\tilde{\mathcal{A}}_k$  are defined as in (7). Since we assumed  $X \rightarrow \mathbb{P}(V)$  to be homologically projectively dual to  $Y \rightarrow \mathbb{P}(V^*)$ , there is an object  $\mathcal{G} \in \mathbf{D}^b(Y \times_{\mathbb{P}(V^*)} \mathcal{H})$  and a semiorthogonal decomposition

$$\mathbf{D}^b(\mathcal{H}) = \left\langle \Phi_{Y \rightarrow \mathcal{H}}^{\mathcal{G}}(\mathbf{D}^b(Y)), \mathcal{A}_1(1) \boxtimes \mathbf{D}^b(\mathbb{P}(V^*)), \dots, \mathcal{A}_{i-1}(i-1) \boxtimes \mathbf{D}^b(\mathbb{P}(V^*)) \right\rangle.$$

Since we assumed the dimensions of  $X_L$  and  $Y_L$  to be as expected we can apply *faithful base change* [Kuz07] to obtain an object  $\mathcal{G}_L \in \mathbf{D}^b(Y_L \times_{\mathbb{P}(L)} \mathcal{H}_L)$  and a semiorthogonal decomposition

$$\mathbf{D}^b(\mathcal{H}_L) = \left\langle \Phi_{Y_L \rightarrow \mathcal{H}_L}^{\mathcal{G}_L}(\mathbf{D}^b(Y_L)), \mathcal{A}_1(1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots, \mathcal{A}_{i-1}(i-1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \right\rangle.$$

It follows that we have the following semiorthogonal decomposition for  $\tilde{\mathcal{H}}$ :

$$(9) \quad \mathbf{D}^b(\tilde{\mathcal{H}}) = \left\langle \mathbf{D}^b(X_L \times \mathbb{P}(L))_{-l+2}, \dots, \mathbf{D}^b(X_L \times \mathbb{P}(L))_{-1}, \right. \\ \left. \tilde{\pi}^* \left( \Phi_{Y_L \rightarrow \mathcal{H}_L}^{\mathcal{G}_L}(\mathbf{D}^b(Y_L)) \right), \pi^* \mathcal{A}_1(1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots, \pi^* \mathcal{A}_{i-1}(i-1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \right\rangle.$$

By Proposition 3.4, left mutation through pieces of the form  $\mathbf{D}^b(X_L \times \mathbb{P}(L))_{-l+k}$  just means tensoring by

$$\mathcal{O}(\tilde{E}) = i_{\tilde{\mathcal{H}}}^* \mathcal{O}(E, 0).$$

It follows exactly as in Proposition 3.4 that one can mutate (9) into (8). Finally, notice that  $\mathbf{D}^b(Y_L)$  is embedded in  $\mathbf{D}^b(\tilde{\mathcal{H}})$  via a Fourier–Mukai transform whose kernel is pushed forward from the fibre product  $\tilde{\mathcal{H}} \times_{\mathbb{P}(L)} Y_L$ . Indeed

$$\tilde{\pi}^* \left( \Phi_{Y_L \rightarrow \mathcal{H}_L}^{\mathcal{G}_L}(-) \right) \otimes \mathcal{O}((l-2)\tilde{E})$$

is given by  $\Phi_{Y_L \rightarrow \tilde{\mathcal{H}}}^{\mathcal{E}_L}$  where  $\mathcal{E}_L \in \mathbf{D}^b(Y_L \times \tilde{\mathcal{H}})$  is

$$\mathcal{E}_L = (\text{id}_{Y_L} \times \tilde{\pi})^* \mathcal{G}_L \otimes p_{\tilde{\mathcal{H}}}^* \mathcal{O}((l-2)\tilde{E}).$$

As  $\tilde{\pi}$  is  $\mathbb{P}(L)$ -linear and by hypothesis  $\mathcal{G}_L$  is pushed forward from the fiber product, one sees that  $\mathcal{E}_L$  is as well pushed forward from  $Y_L \times_{\mathbb{P}(L)} \tilde{\mathcal{H}}$ .  $\square$

**3.3. Base locus with multiplicity.** Recall that there is a purely categorical notion of HP dual. From that point of view then we can define the categorical analogue  $\mathcal{C}_L$  of  $Y_L$  to just be the right orthogonal of the trivial part of  $\mathcal{H}_L$ :

$$\mathcal{C}_L := \left\langle \mathcal{A}_1(1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots, \mathcal{A}_{i-1}(i-1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \right\rangle^\perp.$$

Then the proof of Theorem 3.6 in particular also shows that the categorical HP dual  $\tilde{\mathcal{C}}$  of  $\text{Bl}_{X_L} X \rightarrow \mathbb{P}(L^*)$  is just  $\mathcal{C}_L$ . Note that we are not assuming the existence of a geometric HP dual  $Y$  here. Thus, considering that a categorical HP dual always exists, this story works any time  $X_L$  is smooth and it has the expected codimension. In fact, if we are happy with the purely categorical result we can additionally drop the assumptions on smoothness and correct dimension of  $X_L$ . In this case the category  $\mathcal{C}_L$  won't be "smooth" and so the correct HP dual will turn out to be a categorical resolution of it.

Let  $Z \subset X$  be a smooth subvariety and consider the sublinear system  $L$  of all sections of  $\mathcal{O}_X(1)$  vanishing with order at least  $m \geq 1$  along  $Z$ :

$$L = H^0(\mathcal{I}_Z^{\otimes m}(1)) \subset H^0(\mathcal{O}_X(1)).$$

If we consider the blowup  $\text{Bl}_Z X$  and let the notation be as in the diagram (3), then we can write  $L$  as

$$L = H^0(\pi^* \mathcal{O}_X(1)(-mE)).$$

Note that  $Z$  is just  $X_L$  with the reduced scheme structure and thus we again obtain a regular map  $\text{Bl}_Z X \rightarrow \mathbb{P}(L^*)$ . Also note that for  $m > 1$  the restricted universal hyperplane section  $\mathcal{H}_L$  is not smooth and thus we are not in the hypothesis of Orlov's theorem anymore. However, if we assume that the singularities of  $\mathcal{H}_L$  are nice enough we still have a HPD story:

**Proposition 3.7.** *Let  $Z \subset X$  be a smooth subvariety of codimension  $c$  and  $m \geq 1$ ; set  $a := \lceil \frac{c-1}{m} \rceil$  and assume  $i \leq a$ . Then there is a Lefschetz decomposition of  $\mathbf{D}^b(\text{Bl}_Z X)$  of the form*

$$\mathbf{D}^b(\text{Bl}_Z X) = \left\langle \tilde{\mathcal{A}}_0, \tilde{\mathcal{A}}_1(1), \dots, \tilde{\mathcal{A}}_{a-1}(a-1) \right\rangle,$$



with respect to the line bundle  $\pi^*\mathcal{O}_X(1)(-mE)$ , where the  $\tilde{\mathcal{A}}_k$  are defined as

$$\tilde{\mathcal{A}}_k := \begin{cases} \langle \pi^*\mathcal{A}_k \otimes \mathcal{O}((c-1)E), \mathbf{D}^b(Z)_{-c+1}, \dots, \mathbf{D}^b(Z)_{-c+m} \rangle & \text{if } 0 \leq k \leq i-1, \\ \langle \mathbf{D}^b(Z)_{-c+1}, \dots, \mathbf{D}^b(Z)_{-c+m} \rangle & \text{if } i \leq k \leq a-2, \\ \langle \mathbf{D}^b(Z)_{-c+1}, \dots, \mathbf{D}^b(Z)_{-c+((c-1) \bmod m)} \rangle & \text{if } k = a-1. \end{cases}$$

*Proof.* This is an immediate consequence Proposition 3.4.  $\square$

**Theorem 3.8.** *Let  $\mathrm{Bl}_Z X \rightarrow \mathbb{P}(L^*)$  be as above and equip it with the Lefschetz decomposition of Proposition 3.7. Assume furthermore that  $\mathcal{H}_L$  has only rational singularities. Then the categorical HP dual  $\tilde{\mathcal{C}}$  is a categorical resolution of singularities of  $\mathcal{C}_L$ .*

*Proof.* Recall that we have the two semiorthogonal decompositions

$$(10) \quad \mathbf{D}^b(\tilde{\mathcal{H}}) = \langle \tilde{\mathcal{C}}, \tilde{\mathcal{A}}_1(1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots, \tilde{\mathcal{A}}_{a-1}(a-1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle$$

$$(11) \quad \mathbf{D}^b(\mathcal{H}_L) = \langle \mathcal{C}_L, \mathcal{A}_1(1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots, \mathcal{A}_{i-1}(i-1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle.$$

We apply Proposition 3.4 to mutate (10) into the decomposition

$$\begin{aligned} \mathbf{D}^b(\tilde{\mathcal{H}}) = & \langle \mathbf{D}^b(Z)_{-c+m+1} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots, \mathbf{D}^b(Z)_{-1} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \tilde{\mathcal{C}}' \\ & \pi^*\mathcal{A}_1(1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots, \pi^*\mathcal{A}_{i-1}(i-1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle, \end{aligned}$$

where  $\tilde{\mathcal{C}}' \cong \tilde{\mathcal{C}}$ . We now show that  $\tilde{\mathcal{C}}'$  is a categorical resolution of singularities [Kuz08b] of  $\mathcal{C}_L$ , i.e. there exists a pair of functors

$$\sigma_*: \tilde{\mathcal{C}}' \rightarrow \mathcal{C}_L, \quad \sigma^*: \mathcal{C}_L^{\mathrm{perf}} \rightarrow \tilde{\mathcal{C}}',$$

such that  $\sigma^*$  is left adjoint to  $\sigma_*$  and the unit  $\mathrm{id}_{\mathcal{C}_L^{\mathrm{perf}}} \rightarrow \sigma_*\sigma^*$  is an isomorphism. On the level of the ambient categories we have such a pair given by  $\tilde{\pi}_*$  and  $\tilde{\pi}^*$ . We claim that the restrictions

$$\sigma_* := \tilde{\pi}_*|_{\tilde{\mathcal{C}}'}, \quad \sigma^* := \tilde{\pi}^*|_{\mathcal{C}_L^{\mathrm{perf}}},$$

do the job. Indeed, since we assumed that  $\mathcal{H}_L$  has only rational singularities we have  $\sigma_*\sigma^* \cong \mathrm{id}_{\mathcal{C}_L^{\mathrm{perf}}}$ , and thus we only need to show that  $\sigma_*$  and  $\sigma^*$  have the right codomains. For  $\sigma_*$  this follows from the computation

$$\mathrm{Hom}(i_{\mathcal{H}_L}^*(\mathcal{A}_k(k) \boxtimes \mathbf{D}^b(\mathbb{P}(L))), \tilde{\pi}_*\tilde{\mathcal{C}}') \cong \mathrm{Hom}(i_{\mathcal{H}}^*(\pi^*\mathcal{A}_k(k) \boxtimes \mathbf{D}^b(\mathbb{P}(L))), \tilde{\mathcal{C}}') = 0.$$

For  $\sigma^*$  it follows from

$$\mathrm{Hom}(i_{\mathcal{H}}^*(\pi^*\mathcal{A}_k(k) \boxtimes \mathbf{D}^b(\mathbb{P}(L))), \tilde{\pi}^*\mathcal{C}_L^{\mathrm{perf}}) \cong \mathrm{Hom}(i_{\mathcal{H}_L}^*(\mathcal{A}_k(k) \boxtimes \mathbf{D}^b(\mathbb{P}(L))), \mathcal{C}_L^{\mathrm{perf}}) = 0,$$

and

$$\begin{aligned} \mathrm{Hom}(\tilde{\pi}^*\mathcal{C}_L^{\mathrm{perf}}, i_{\mathcal{H}}^*(\mathbf{D}^b(Z)_k \boxtimes \mathbf{D}^b(\mathbb{P}(L)))) & \cong \mathrm{Hom}(\tilde{\pi}^*\mathcal{C}_L^{\mathrm{perf}}, \tilde{j}_*\tilde{p}^*\mathbf{D}^b(Z \times \mathbb{P}(L)) \otimes \mathcal{O}(-k\tilde{E})) \\ & \cong \mathrm{Hom}(\tilde{p}^*\tilde{i}^*\mathcal{C}_L^{\mathrm{perf}}, \tilde{p}^*\mathbf{D}^b(Z \times \mathbb{P}(L)) \otimes \mathcal{O}_{\tilde{E}}(k)) \\ & \cong \mathrm{Hom}(\tilde{i}^*\mathcal{C}_L^{\mathrm{perf}}, \mathbf{D}^b(Z \times \mathbb{P}(L)) \otimes \tilde{p}_*\mathcal{O}_{\tilde{E}}(k)) \\ & = 0, \end{aligned}$$

for  $-c+1 \leq k \leq -1$ .  $\square$

## 4. TWO EXAMPLES

A natural question to ask at this point is whether in the case of base locus with multiplicity we can say something more about the categorical HP dual  $\tilde{\mathcal{C}}$  instead of just that it is a categorical resolution of singularities of  $\mathcal{C}_L$ . For example: could  $\tilde{\mathcal{C}}$  be geometric? We will present two examples showing that this is sometimes the case, depending on the choice of linear system  $L$ . More precisely the starting point will be the degree 3 Veronese embedding  $\mathbb{P}^5 \hookrightarrow \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^5}(3))^*)$  equipped with the standard Lefschetz decomposition of  $\mathbb{P}^5$  with respect to  $\mathcal{O}_{\mathbb{P}^5}(3)$ . The HP dual in this case is a noncommutative K3-fibration. Both examples are essentially applications of Kuznetsov's results on cubic fourfolds [Kuz10].

**4.1. First example.** We start with an example where the above process yields geometric HP duality. We will take  $L \subset H^0(\mathcal{O}_{\mathbb{P}^5}(3))$  to be the linear system of all cubic fourfolds that are singular at a fixed point  $P \in \mathbb{P}^5$ . In particular then on  $\text{Bl}_P \mathbb{P}^5$  we have

$$L \cong H^0(\pi^* \mathcal{O}_{\mathbb{P}^5}(3)(-2E)),$$

and it is base point free. Finally by Proposition 3.7 we have a rectangular Lefschetz decomposition of  $\text{Bl}_P \mathbb{P}^5$  with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^5}(3)(-2E)$  given by

$$(12) \quad \mathbf{D}^b(\text{Bl}_P \mathbb{P}^5) = \langle \tilde{\mathcal{A}}, \tilde{\mathcal{A}}(1) \rangle,$$

where

$$\tilde{\mathcal{A}} = \langle \pi^* \mathcal{O}_{\mathbb{P}^5}(-6)(4E), \pi^* \mathcal{O}_{\mathbb{P}^5}(-5)(4E), \pi^* \mathcal{O}_{\mathbb{P}^5}(-4)(4E), \mathbf{D}^b(P)_{-4}, \mathbf{D}^b(P)_{-3} \rangle.$$

**Proposition 4.1.** *The HP dual of  $\text{Bl}_P \mathbb{P}^5 \rightarrow \mathbb{P}(L^*)$  with respect to the above Lefschetz decomposition is a complete intersection  $\check{X}$  of a universal  $(2, 1)$  and a universal  $(3, 1)$  divisor in  $\mathbb{P}^4 \times \mathbb{P}(L)$ . In particular  $\check{X} \rightarrow \mathbb{P}(L)$  is generically a K3-fibration.*

To prove this proposition we follow closely Calabrese and Thomas [CT15]. Note first that linear projection away from  $P$  defines a rational map  $\mathbb{P}^5 \dashrightarrow \mathbb{P}^4$  with indeterminacy locus  $P$  and thus induces a regular map  $\phi: \text{Bl}_P \mathbb{P}^5 \rightarrow \mathbb{P}^4$  which exhibits  $\text{Bl}_P \mathbb{P}^5$  as a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^4$  given by

$$\phi: \mathbb{P}(\mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}) \rightarrow \mathbb{P}^4.$$

This carries a tautological line bundle which we denote by  $\mathcal{O}_\phi(-1)$ . From this point of view the exceptional divisor  $E$  is cut out by a section of  $\mathcal{O}_\phi(1)$  which is the image of the section  $(0, 1) \in H^0(\mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4})$  under the tautological isomorphism

$$H^0(\mathcal{O}_\phi(1)) \cong H^0(\mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4}),$$

and thus we obtain

$$\mathcal{O}_\phi(1) \cong \mathcal{O}(E).$$

Using the fact that  $\phi^* \mathcal{O}_{\mathbb{P}^4}(1) \cong \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E)$  and the projection formula we can now compute

$$\phi_* \pi^* \mathcal{O}_{\mathbb{P}^5}(3)(-2E) \cong \mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(3).$$

In particular we have

$$H^0(\pi^* \mathcal{O}_{\mathbb{P}^5}(3)(-2E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(1)) \cong H^0(\mathcal{O}_{\mathbb{P}^4 \times \mathbb{P}(L)}(2, 1)) \oplus H^0(\mathcal{O}_{\mathbb{P}^4 \times \mathbb{P}(L)}(3, 1)),$$

and thus the tautological section which cuts out the universal hyperplane section  $\tilde{\mathcal{H}} \subset \text{Bl}_P \mathbb{P}^5 \times \mathbb{P}(L)$  induces a section cutting out  $\check{X} \subset \mathbb{P}^4 \times \mathbb{P}(L)$ . An argument by Calabrese and Thomas [CT15, Lemma 4.3] shows that we have an isomorphism

$$(13) \quad \tilde{\mathcal{H}} \cong \text{Bl}_{\check{X}}(\mathbb{P}^4 \times \mathbb{P}(L)),$$

where the projection  $\tilde{\pi}: \tilde{\mathcal{H}} \rightarrow \mathbb{P}^4 \times \mathbb{P}(L)$  is given by

$$\tilde{\pi} = (\phi \times \text{id}_{\mathbb{P}(L)}) \circ i_{\tilde{\mathcal{H}}}.$$

We will denote the exceptional divisor of this blowup by  $\check{E}$ . Recall from the previous sections that we also have an isomorphism

$$\tilde{\mathcal{H}} \cong \mathrm{Bl}_{P \times \mathbb{P}(L)} \mathcal{H}_L.$$

Looking at the defining equations one can compute explicitly that the exceptional divisor  $\check{E}$  of this blowup gets mapped to the  $(2, 1)$  divisor containing  $\check{X}$  under the composition of the isomorphism (13) and the projection  $\tilde{\pi}$ . In fact  $\check{E}$  is the proper transform of this divisor and so we compute

$$\begin{aligned} \mathcal{O}(\check{E}) &\cong \tilde{\pi}^* \mathcal{O}_{\mathbb{P}^4 \times \mathbb{P}(L)}(2, 1)(-\check{E}) \\ &\cong i_{\tilde{\mathcal{H}}}^* (\pi^* \mathcal{O}_{\mathbb{P}^5}(2)(-2E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(1))(-\check{E}). \end{aligned}$$

Using the fact that  $\mathcal{O}(\check{E}) \cong i_{\tilde{\mathcal{H}}}^* (\mathcal{O}(E) \boxtimes \mathcal{O}_{\mathbb{P}(L)})$  we can rewrite this as

$$\mathcal{O}(\check{E}) \cong i_{\tilde{\mathcal{H}}}^* (\pi^* \mathcal{O}_{\mathbb{P}^5}(2)(-3E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(1)).$$

To summarise, we have the following commutative diagram:

$$\begin{array}{ccccccc} E \times \mathbb{P}(L) & \xrightarrow{j \times \mathrm{id}} & \mathrm{Bl}_P \mathbb{P}^5 \times \mathbb{P}(L) & \xleftarrow{i_{\tilde{\mathcal{H}}}} & \tilde{\mathcal{H}} & & \\ \uparrow i_{\tilde{\mathcal{H}}} & & \uparrow i_{\tilde{\mathcal{H}}} & & \downarrow \cong & & \\ \check{E} & \hookrightarrow & \mathrm{Bl}_{P \times \mathbb{P}(L)} \mathcal{H}_L & \xrightarrow{\cong} & \mathrm{Bl}_{\check{X}}(\mathbb{P}^4 \times \mathbb{P}(L)) & \xleftarrow{j} & \check{E} \\ \downarrow & & \downarrow & & \downarrow \tilde{\pi} & & \downarrow \check{p} \\ P \times \mathbb{P}(L) & \hookrightarrow & \mathcal{H}_L & & \mathbb{P}^4 \times \mathbb{P}(L) & \xleftarrow{i} & \check{X} \end{array}$$

We are now ready to start imitating Kuznetsov's mutations [Kuz10] to prove the proposition. First note that after applying an overall twist by  $\pi^* \mathcal{O}_{\mathbb{P}^5}(3)(-2E)$  to the semiorthogonal decomposition of  $\mathbf{D}^b(\mathcal{H})$  coming from (12) we get:

$$\begin{aligned} \mathbf{D}^b(\tilde{\mathcal{H}}) &= \left\langle j_* \mathcal{O}_E \otimes \pi^* \mathcal{O}_{\mathbb{P}^5}(-2)(2E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), j_* \mathcal{O}_E \otimes \pi^* \mathcal{O}_{\mathbb{P}^5}(-1)(E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \right. \\ &\quad \left. \tilde{\mathcal{C}}, \pi^* \mathcal{O}_{\mathbb{P}^5} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \pi^* \mathcal{O}_{\mathbb{P}^5}(1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \pi^* \mathcal{O}_{\mathbb{P}^5}(2) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \right\rangle. \end{aligned}$$

Note that in the case of  $\mathbf{D}^b(P)_{-4}$  we chose to write the structure sheaf of the point as  $i^* \mathcal{O}_{\mathbb{P}^5}(-5)$ , whereas in the case of  $\mathbf{D}^b(P)_{-3}$  we chose to write it as  $i^* \mathcal{O}_{\mathbb{P}^5}(-4)$ ; the reasons for this will become clear later. Let us now show that  $\tilde{\mathcal{C}}$  is equivalent to  $\mathbf{D}^b(\check{X})$ . To do this we begin with the semiorthogonal decomposition of  $\tilde{\mathcal{H}}$  coming from the isomorphism (13) and Orlov's theorem:

$$\begin{aligned} \mathbf{D}^b(\tilde{\mathcal{H}}) &= \left\langle \Phi(\mathbf{D}^b(\check{X})), \pi^* \mathcal{O}_{\mathbb{P}^5}(-3)(3E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \right. \\ &\quad \left. \pi^* \mathcal{O}_{\mathbb{P}^5}(-2)(2E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots, \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \right\rangle. \end{aligned}$$

Here  $\Phi$  is the embedding given by  $\Phi(-) = \check{j}_* \check{p}^*(-) \otimes \mathcal{O}(\check{E})$  and we chose to decompose  $\mathbf{D}^b(\mathbb{P}^4)$  as

$$\mathbf{D}^b(\mathbb{P}^4) = \langle \mathcal{O}_{\mathbb{P}^4}(-3), \mathcal{O}_{\mathbb{P}^4}(-2), \dots, \mathcal{O}_{\mathbb{P}^4}(1) \rangle.$$

In the above semiorthogonal decompositions of  $\mathbf{D}^b(\tilde{\mathcal{H}})$  and all that follow we will usually suppress writing the implicit restriction  $i_{\tilde{\mathcal{H}}}^*$  since it is always fully faithful on all the pieces that appear and we implicitly keep applying the following lemma:

**Lemma 4.2.** *Let  $\mathcal{A} \subset \mathbf{D}^b(X)$  be an admissible full triangulated subcategory. If  $\mathcal{F} \in \mathcal{A}$  is an exceptional object, then for any  $\mathcal{G} \in \mathcal{A}$  we have isomorphisms*

$$\begin{aligned} \mathbb{L}_{i_{\mathcal{A}}(\mathcal{F})}(i_{\mathcal{A}}(\mathcal{G})) &\cong i_{\mathcal{A}}(\mathbb{L}_{\mathcal{F}}(\mathcal{G})), \\ \mathbb{R}_{i_{\mathcal{A}}(\mathcal{F})}(i_{\mathcal{A}}(\mathcal{G})) &\cong i_{\mathcal{A}}(\mathbb{R}_{\mathcal{F}}(\mathcal{G})), \end{aligned}$$

where  $i_{\mathcal{A}}: \mathcal{A} \hookrightarrow \mathcal{T}$  denotes the inclusion functor.

*Proof.* This is just a simple consequence of the fact that one can write mutation through an exceptional object explicitly. E.g. in the case of left mutation we compute

$$\begin{aligned} \mathbb{L}_{i_{\mathcal{A}}(\mathcal{F})}(i_{\mathcal{A}}(\mathcal{G})) &= \text{Cone}(i_{\mathcal{A}}(\mathbf{R}\text{Hom}(\mathcal{F}, i_{\mathcal{A}}^!(i_{\mathcal{A}}(\mathcal{G})))) \otimes \mathcal{F} \rightarrow i_{\mathcal{A}}(\mathcal{G})) \\ &= i_{\mathcal{A}}(\text{Cone}(\mathbf{R}\text{Hom}(\mathcal{F}, \mathcal{G}) \otimes \mathcal{F} \rightarrow \mathcal{G})) \\ &= i_{\mathcal{A}}(\mathbb{L}_{\mathcal{F}}(\mathcal{G})). \end{aligned}$$

The case of right mutation is analogous.  $\square$

*Step 1.* We mutate the first three pieces after  $\Phi(\mathbf{D}^b(\check{X}))$  all the way to the left. To do this we apply Proposition 3.4 to obtain

$$\mathbb{L}_{\Phi(\mathbf{D}^b(\check{X}))}(\pi^* \mathcal{F}) = \check{\pi}^* \mathcal{F}(\check{E}).$$

Using the previous expression for  $\mathcal{O}(\check{E})$  we obtain the following semiorthogonal decomposition after the mutation:

$$\begin{aligned} \mathbf{D}^b(\tilde{\mathcal{H}}) &= \left\langle \pi^* \mathcal{O}_{\mathbb{P}^5}(-1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \pi^* \mathcal{O}_{\mathbb{P}^5}(-E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \right. \\ &\quad \left. \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-2E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \Phi(\mathbf{D}^b(\check{X})), \pi^* \mathcal{O}_{\mathbb{P}^5} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \right\rangle. \end{aligned}$$

*Step 2.* We mutate  $\Phi(\mathbf{D}^b(\check{X}))$  all the way to the right. All we need to remark at this point is that  $\Phi$  is given by the Fourier–Mukai transform whose kernel is

$$(\check{p} \times \check{j})_* \mathcal{O}_{\check{E}}(-1),$$

which is of course supported on the fibre product  $\check{X} \times_{\mathbb{P}(L)} \tilde{\mathcal{H}}$ . We can write the result of the mutation in terms of a modified embedding

$$\Phi' = \mathbb{R}_{\pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E) \boxtimes \mathbf{D}^b(\mathbb{P}(L))} \circ \mathbb{R}_{\pi^* \mathcal{O}_{\mathbb{P}^5} \boxtimes \mathbf{D}^b(\mathbb{P}(L))} \circ \Phi,$$

and we just want to make sure that  $\Phi'$  can still be written as a Fourier–Mukai transform whose kernel is supported on  $\check{X} \times_{\mathbb{P}(L)} \tilde{\mathcal{H}}$ . This fact is an immediate consequence of the following two lemmas:

**Lemma 4.3.** *Let  $X, Y, Z$  be smooth projective varieties equipped with regular maps to a smooth projective variety  $B$  and assume that either one of the projections  $X \times_B Y \rightarrow Y$  or  $Y \times_B Z \rightarrow Y$  is flat and  $Y$  is proper over  $B$ . Given  $\mathcal{F} \in \mathbf{D}^b(X \times_B Y)$  and  $\mathcal{G} \in \mathbf{D}^b(Y \times_B Z)$ , if either one of the kernels is perfect, the relative convolution*

$$\pi_{X,Z}^B (\pi_{X,Y}^{B,*} \mathcal{F} \otimes \pi_{Y,Z}^{B,*} \mathcal{G})$$

*gives an isomorphism*

$$\Phi_{Y \rightarrow Z}^{B, \mathcal{G}} \circ \Phi_{X \rightarrow Y}^{B, \mathcal{F}} \cong \Phi_{X \rightarrow Z}^{B, \mathcal{F} * \mathcal{G}},$$

*with  $\mathcal{F} * \mathcal{G} \in \mathbf{D}^b(X \times_B Z)$ .*

*Proof.* This is just a matter of looking at the diagram of the corresponding proposition in Huybrechts' book [Huy06, Proposition 5.10] and noting that with the flatness assumption we get an exact Cartesian square [Kuz07, Section 2.6] and everything else generalises from products to fibre products. Observe that if one prefers to work with absolute integral functors, one just notices that

$$\Phi_{X \rightarrow Y}^{B, \mathcal{F}} \cong \Phi_{X \rightarrow Y}^{j_{X,Y,*} \mathcal{F}}$$

where  $j_{X,Y}: X \times_B Y \rightarrow X \times Y$ .  $\square$

**Lemma 4.4.** *Let  $i_Y: Y \hookrightarrow X \times B$  be a closed immersion where all the varieties are smooth and projective, and assume that there is an exceptional object  $\mathcal{F} \in \mathbf{D}^b(X)$  such that the derived pullback functor  $i_Y^*$  is fully faithful on  $\langle \mathcal{F} \rangle \boxtimes \mathbf{D}^b(B)$ . Then both left and right mutation through  $i_Y^*(\mathcal{F} \boxtimes \mathbf{D}^b(B))$  in  $\mathbf{D}^b(Y)$  can be written as a Fourier–Mukai transform whose kernel is supported on  $Y \times_B Y$ .*

*Proof.* Note first that the left adjoint  $i_{Y,!}$  of the derived pullback  $i_Y^*$  is given by

$$i_{Y,!}(-) = i_{Y,*}(- \otimes \Lambda^c \mathcal{N}[-c]),$$

where  $\mathcal{N}$  is the normal bundle of  $Y$  in  $X \times B$  and  $c$  is the codimension. Now we can say that left and right mutation through  $i_Y^*(\mathcal{F} \boxtimes \mathbf{D}^b(B))$  in  $\mathbf{D}^b(Y)$  are given by the Fourier–Mukai transforms with kernels

$$\begin{aligned} \text{Cone}(\mathcal{K}_{\mathbb{L}} \rightarrow \mathcal{O}_{\Delta_Y}), \\ \text{Cone}(\mathcal{O}_{\Delta_Y} \rightarrow \mathcal{K}_{\mathbb{R}})[-1], \end{aligned}$$

respectively, where  $\mathcal{K}_{\mathbb{L}}$  and  $\mathcal{K}_{\mathbb{R}}$  are the convolutions

$$\begin{aligned} \mathcal{K}_{\mathbb{L}} &= ((\text{id}_Y \times i_Y)_* \mathcal{O}_Y) * (\mathcal{F}^\vee \boxtimes \mathcal{F} \boxtimes \mathcal{O}_{\Delta_B}) * ((i_Y \times \text{id}_Y)_* \mathcal{O}_Y), \\ \mathcal{K}_{\mathbb{R}} &= ((\text{id}_Y \times i_Y)_* \Lambda^c \mathcal{N}[-c]) * ((\mathcal{F}^\vee \otimes \omega_X[\dim X]) \boxtimes \mathcal{F} \boxtimes \mathcal{O}_{\Delta_B}) * ((i_Y \times \text{id}_Y)_* \mathcal{O}_Y). \end{aligned}$$

Observe that the graph map  $(\text{id}_Y \times_B i_Y): Y \rightarrow Y \times_B (X \times B)$  is a regular embedding, so by Lemma 4.3 it is possible to consider the relative convolution. We can actually compute the convoluted kernels explicitly, getting:

$$\begin{aligned} \mathcal{K}_{\mathbb{L}} &= (i_Y \times_B i_Y)^*(\mathcal{F}^\vee \boxtimes \mathcal{F} \boxtimes \mathcal{O}_{\Delta_B}), \\ \mathcal{K}_{\mathbb{R}} &= (i_Y \times_B i_Y)^*((\mathcal{F}^\vee \otimes \omega_X[\dim X]) \boxtimes \mathcal{F} \boxtimes \mathcal{O}_{\Delta_B}) \otimes \pi_Y^* \Lambda^c \mathcal{N}[-c], \end{aligned}$$

where we denoted by  $\pi_Y: Y \times_B Y \rightarrow Y$  the projection onto the second factor.  $\square$

Looking at the explicit expression of  $\text{Cone}(\mathcal{O}_{\Delta_Y} \rightarrow \mathcal{K}_{\mathbb{R}})[-1]$  we see that for  $\mathcal{F}$  locally free, the mutations through  $i_{\mathcal{H}}^*(\mathcal{F} \boxtimes \mathbf{D}^b(\mathbb{P}(L)))$  can be written as relative Fourier–Mukai transforms whose kernels are actually perfect. We thus conclude that after mutating we have the semiorthogonal decomposition

$$\begin{aligned} \mathbf{D}^b(\tilde{\mathcal{H}}) &= \left\langle \pi^* \mathcal{O}_{\mathbb{P}^5}(-1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \pi^* \mathcal{O}_{\mathbb{P}^5}(-E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \right. \\ &\quad \left. \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-2E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \pi^* \mathcal{O}_{\mathbb{P}^5} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \Phi'(\mathbf{D}^b(\check{X})) \right\rangle. \end{aligned}$$

*Step 3.* We now transpose the third and fourth piece of the semiorthogonal decomposition since they are completely orthogonal. Indeed we have

$$\text{Ext}_{\mathbf{D}^b(\text{Bl}_P \mathbb{P}^5)}^\bullet(\pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-2E), \pi^* \mathcal{O}_{\mathbb{P}^5}) = 0$$

by Proposition 3.4 and thus from the Künneth formula it follows that our pieces are completely orthogonal. The semiorthogonal decomposition after this step is now

$$\begin{aligned} \mathbf{D}^b(\tilde{\mathcal{H}}) &= \left\langle \pi^* \mathcal{O}_{\mathbb{P}^5}(-1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \pi^* \mathcal{O}_{\mathbb{P}^5}(-E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \right. \\ &\quad \left. \pi^* \mathcal{O}_{\mathbb{P}^5} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-2E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \Phi'(\mathbf{D}^b(\check{X})) \right\rangle. \end{aligned}$$

*Step 4.* We now right mutate the second piece through the third one and the fourth one through the fifth one. We will only explain the first case since the second one works in exactly the same way. For that we will decompose  $\mathbf{D}^b(\mathbb{P}(L))$  in the usual way so that we’re effectively mutating exceptional objects. Again by Proposition 3.4 and the Künneth formula we have for  $0 \leq k' \leq k \leq l$ :

$$\dim \text{Ext}^p(\pi^* \mathcal{O}_{\mathbb{P}^5}(-E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k), \pi^* \mathcal{O}_{\mathbb{P}^5} \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k')) = \begin{cases} 1 & \text{if } p = 0 \text{ and } k = k', \\ 0 & \text{else.} \end{cases}$$

The first implication of this is that we can just swap pieces until we are in the situation of  $k = k'$ . In that case then we have the distinguished triangle

$$\mathbb{R}_{\pi^* \mathcal{O}_{\mathbb{P}^5}}(\pi^* \mathcal{O}_{\mathbb{P}^5}(-E)) \rightarrow \pi^* \mathcal{O}_{\mathbb{P}^5}(-E) \rightarrow \pi^* \mathcal{O}_{\mathbb{P}^5} \rightarrow \cdots,$$

where we omitted the  $(-) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)$  in every term. But we also have the short exact sequence

$$0 \rightarrow \pi^* \mathcal{O}_{\mathbb{P}^5}(-E) \rightarrow \pi^* \mathcal{O}_{\mathbb{P}^5} \rightarrow j_* \mathcal{O}_E \rightarrow 0,$$

and thus after shifting and comparing we obtain

$$\mathbb{R}_{\pi^* \mathcal{O}_{\mathbb{P}^5} \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)}(\pi^* \mathcal{O}_{\mathbb{P}^5}(-E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)) \cong j_* \mathcal{O}_E[-1] \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k).$$

With the same argument for the second case we obtain

$$\begin{aligned} \mathbb{R}_{\pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)}(\pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-2E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)) &\cong \\ &\cong j_* \mathcal{O}_E \otimes \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E)[-1] \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k). \end{aligned}$$

Finally we note that cases where  $k' > k$  only arise once a piece has been properly mutated and then we again have complete orthogonality and can finish with a series of swaps. The semiorthogonal decomposition that we finally obtain is thus

$$\begin{aligned} \mathbf{D}^b(\tilde{\mathcal{H}}) = & \left\langle \pi^* \mathcal{O}_{\mathbb{P}^5}(-1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \pi^* \mathcal{O}_{\mathbb{P}^5} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), j_* \mathcal{O}_E \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \right. \\ & \left. \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), j_* \mathcal{O}_E \otimes \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \Phi'(\mathbf{D}^b(\check{X})) \right\rangle. \end{aligned}$$

*Step 5.* We now left mutate the fourth piece through the third one. For that we again decompose  $\mathbf{D}^b(\mathbb{P}(L))$  as usual. By Proposition 3.4, the Künneth formula and the fact that  $j$  is a divisorial embedding we now have for  $0 \leq k' \leq k \leq l$ :

$$\dim \operatorname{Ext}^p(j_* \mathcal{O}_E \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k), \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k')) = \begin{cases} 1 & \text{if } p = 1 \text{ and } k = k', \\ 0 & \text{else.} \end{cases}$$

It follows again that we can just swap pieces until we are in the case of  $k = k'$  and we have the distinguished triangle

$$j_* \mathcal{O}_E[-1] \rightarrow \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E) \rightarrow \mathbb{L}_{j_* \mathcal{O}_E}(\pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E)) \rightarrow \cdots,$$

where we again omitted  $(-) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)$  in every term. Since we are blowing up a point we have the short exact sequence

$$0 \rightarrow \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E) \rightarrow \pi^* \mathcal{O}_{\mathbb{P}^5}(1) \rightarrow j_* \mathcal{O}_E \rightarrow 0.$$

Thus after shifting and comparing we obtain

$$\mathbb{L}_{j_* \mathcal{O}_E \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)}(\pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)) \cong \pi^* \mathcal{O}_{\mathbb{P}^5}(1) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k).$$

Hence we now have the semiorthogonal decomposition

$$\begin{aligned} \mathbf{D}^b(\tilde{\mathcal{H}}) = & \left\langle \pi^* \mathcal{O}_{\mathbb{P}^5}(-1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \pi^* \mathcal{O}_{\mathbb{P}^5} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \pi^* \mathcal{O}_{\mathbb{P}^5}(1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \right. \\ & \left. j_* \mathcal{O}_E \boxtimes \mathbf{D}^b(\mathbb{P}(L)), j_* \mathcal{O}_E \otimes \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \Phi'(\mathbf{D}^b(\check{X})) \right\rangle. \end{aligned}$$

*Step 6.* Finally we mutate the rightmost three pieces all the way to left. For this we note that the canonical bundle of  $\tilde{\mathcal{H}}$  is

$$\omega_{\tilde{\mathcal{H}}} = i_{\tilde{\mathcal{H}}}^* (\pi^* \mathcal{O}_{\mathbb{P}^5}(-3)(2E) \boxtimes \omega_{\mathbb{P}(L)}(1)),$$

and thus after an additional twist by  $i_{\tilde{\mathcal{H}}}^* (\pi^* \mathcal{O}_{\mathbb{P}^5}(1) \boxtimes \mathcal{O}_{\mathbb{P}(L)})$  we obtain

$$\begin{aligned} \mathbf{D}^b(\tilde{\mathcal{H}}) = & \left\langle j_* \mathcal{O}_E \otimes \pi^* \mathcal{O}_{\mathbb{P}^5}(-2)(2E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), j_* \mathcal{O}_E \otimes \pi^* \mathcal{O}_{\mathbb{P}^5}(-1)(E) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \right. \\ & \left. \Phi''(\mathbf{D}^b(\check{X})), \pi^* \mathcal{O}_{\mathbb{P}^5} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \pi^* \mathcal{O}_{\mathbb{P}^5}(1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \pi^* \mathcal{O}_{\mathbb{P}^5}(2) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \right\rangle. \end{aligned}$$

Note that  $\Phi''$  is just  $\Phi'$  composed with tensoring and thus still a Fourier–Mukai transform whose kernel is supported on  $\check{X} \times_{\mathbb{P}(L)} \tilde{\mathcal{H}}$ ; hence we are done.

*Remark 4.5.* Note that by Theorem 3.8 we see that  $\mathbf{D}^b(\check{X})$  is a categorical resolution of singularities of the noncommutative K3-fibration  $\mathcal{C}_L \rightarrow \mathbb{P}(L)$ . This is basically a family version of Kuznetsov’s result on singular cubic fourfolds [Kuz10]. By taking a generic pencil we also recover Calabrese and Thomas’ example of derived equivalent Calabi–Yau threefolds [CT15].

**4.2. Second example.** Consider now the linear system

$$L = H^0(\mathcal{I}_{\mathbb{P}(W)}(3)) \subset H^0(\mathcal{O}_{\mathbb{P}^5}(3))$$

of cubic fourfolds containing a plane  $\mathbb{P}(W)$ . Looking at the blowup  $\mathrm{Bl}_{\mathbb{P}(W)} \mathbb{P}^5$ , we can identify  $L$  with the complete linear system  $H^0(\pi^* \mathcal{O}_{\mathbb{P}^5}(3)(-E))$ , where as usual  $\pi$  denotes the projection from the blow up. The line bundle  $\pi^* \mathcal{O}_{\mathbb{P}^5}(3)(-E)$  has no base locus [CT15, Lemma 3.5] so we get a regular map

$$\begin{array}{ccc} \mathrm{Bl}_{\mathbb{P}(W)} \mathbb{P}^5 & & \\ \downarrow \pi & \searrow f & \\ \mathbb{P}^5 & \dashrightarrow & \mathbb{P}(L^*) \end{array}$$

By Proposition 3.4 the following is a Lefschetz decomposition for  $\mathrm{Bl}_{\mathbb{P}(W)} \mathbb{P}^5$  with respect to the line bundle  $\pi^* \mathcal{O}(3)(-E)$ :

$$\mathbf{D}^b(\mathrm{Bl}_{\mathbb{P}(W)} \mathbb{P}^5) = \langle \tilde{\mathcal{A}}, \tilde{\mathcal{A}}(1) \rangle,$$

where we recall that

$$\tilde{\mathcal{A}} := \langle \pi^* \mathcal{A} \otimes \mathcal{O}(2E), \mathbf{D}^b(\mathbb{P}(W))_{-2} \rangle,$$

$\mathcal{A}$  being the first component in the rectangular Lefschetz decomposition of  $\mathbf{D}^b(\mathbb{P}^5)$  with respect to  $\mathcal{O}_{\mathbb{P}^5}(3)$ . Write  $\mathbb{P}^5 = \mathbb{P}(W \oplus W')$ .

**Proposition 4.6.** *The HP dual of  $\mathrm{Bl}_{\mathbb{P}(W)} \mathbb{P}^5 \rightarrow \mathbb{P}(L^*)$  with respect to the above Lefschetz decomposition is the noncommutative variety  $(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0)$ , where  $\mathcal{C}_0$  is the even Clifford algebra sheaf on  $\mathbb{P}(W') \times \mathbb{P}(L)$  corresponding to the quadric fibration  $\tilde{\pi}$ :*

$$\begin{array}{ccc} \tilde{\mathcal{H}} & \xrightarrow{i_{\tilde{\mathcal{H}}}} & \mathbb{P}(\underline{W}(1) \oplus \mathcal{O}_{\mathbb{P}(W')} \boxtimes \mathcal{O}_{\mathbb{P}(L)}(1)) \\ & \searrow \tilde{\pi} & \downarrow \phi \times \mathrm{id}_{\mathbb{P}(L)} \\ & & \mathbb{P}(W') \times \mathbb{P}(L) \end{array}$$

The claim indeed follows as a direct generalisation of [CT15, Section 3], [Kuz10, Section 4]. Let’s start explaining why  $\tilde{\pi}$  is a quadric fibration. The linear projection  $\mathbb{P}^5 \dashrightarrow \mathbb{P}(W')$  induces a regular map

$$\mathrm{Bl}_{\mathbb{P}(W)} \mathbb{P}^5 \xrightarrow{\phi} \mathbb{P}(W').$$

As pointed out in [CT15],  $\phi$  is the  $\mathbb{P}^3$ -bundle given by the projective completion  $\mathbb{P}(W \otimes \mathcal{O}_{\mathbb{P}(W')}(1) \oplus \mathcal{O}_{\mathbb{P}(W')})$  of  $W \otimes \mathcal{O}_{\mathbb{P}(W')}(1)$  over  $\mathbb{P}(W')$ . Computing the Grothendieck line bundle one gets  $\mathcal{O}_\phi(1) \cong \mathcal{O}_{\tilde{\mathbb{P}}^5}(E)$ ; this computation together with the canonical relation

$$(14) \quad \pi^* \mathcal{O}_{\mathbb{P}^5}(1)(-E) = \phi^* \mathcal{O}_{\mathbb{P}(W')}(1)$$

gives

$$\pi^* \mathcal{O}_{\mathbb{P}^5}(3)(-E) = \mathcal{O}_\phi(2) \otimes \phi^* \mathcal{O}_{\mathbb{P}(W')}(3).$$

This means that the universal family  $\tilde{\mathcal{H}}$  that we know is cut out by the tautological section of  $\pi^* \mathcal{O}_{\mathbb{P}^5}(3)(-E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(1)$  is an element of the linear system

$$|\mathcal{O}_\phi(2) \otimes \phi^* \mathcal{O}_{\mathbb{P}(W')}(3) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(1)|,$$

namely, it is cut out by a section of  $S^2(\underline{W}^* \oplus \mathcal{O}_{\mathbb{P}(W')}(1)) \otimes \mathcal{O}_{\mathbb{P}(W') \times \mathbb{P}(L)}(1, 1)$  (we refer again to [CT15, Section 3] for the details of the computation). By [Kuz08a, Theorem 4.2] there exists a semiorthogonal decomposition

$$(15) \quad \mathbf{D}^b(\tilde{\mathcal{H}}) = \langle \Phi(\mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L)), \mathcal{C}_0), \tilde{\pi}^* \mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L)) \otimes \mathcal{O}_{\tilde{\pi}}(1), \\ \tilde{\pi}^* \mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L)) \otimes \mathcal{O}_{\tilde{\pi}}(2) \rangle$$

where we are denoting with  $\mathcal{O}_{\tilde{\pi}}(1)$  the restriction to  $\tilde{\mathcal{H}}$  of the tautological line bundle on  $\mathbb{P}(\underline{W}(1) \oplus \mathcal{O}_{\mathbb{P}(W')} \boxtimes \mathcal{O}_{\mathbb{P}(L)}(1))$ . Note that

$$\mathcal{O}_{\tilde{\pi}}(1) \cong i_{\tilde{\mathcal{H}}}^* (\mathcal{O}_\phi(1) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(1)) \cong i_{\tilde{\mathcal{H}}}^* (\mathcal{O}_{\tilde{\mathbb{P}}^5}(E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(1)).$$

After an overall twist by  $\mathcal{O}(-E, 0)$  we can rewrite (15) in the following way:

$$(16) \quad \mathbf{D}^b(\tilde{\mathcal{H}}) = \langle \Phi'(\mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0)), \phi^* \mathbf{D}^b(\mathbb{P}(W') \boxtimes \mathbf{D}^b(\mathbb{P}(L))), \\ \phi^* \mathbf{D}^b(\mathbb{P}(W')(H) \boxtimes \mathbf{D}^b(\mathbb{P}(L))) \rangle,$$

where we denote by  $H$  the twisting by  $\pi^* \mathcal{O}_{\mathbb{P}^5}(1)$  and by  $\Phi'$  the faithful embedding given by  $\Phi'(-) = \Phi(-) \otimes \mathcal{O}(-E, 0)$ . In (16) and in all that follows, we drop the  $i_{\tilde{\mathcal{H}}}^*$  to simplify the notation, as it is fully faithful anyway. Before going into mutations, we rewrite (16) in a more explicit way. Replace the first instance of  $\mathbf{D}^b(\mathbb{P}(W'))$  with the exceptional collection  $(\mathcal{O}_{\mathbb{P}(W')}(-1), \mathcal{O}_{\mathbb{P}(W')}, \mathcal{O}_{\mathbb{P}(W')}(1))$  and the second one with the exceptional collection  $(\mathcal{O}_{\mathbb{P}(W')}, \mathcal{O}_{\mathbb{P}(W')}(1), \mathcal{O}_{\mathbb{P}(W')}(2))$  and denote by  $\mathcal{O}(h)$  the twisting by  $\phi^* \mathcal{O}_{\mathbb{P}(W')}(1)$ . We get

$$(17) \quad \mathbf{D}^b(\tilde{\mathcal{H}}) = \langle \Phi'(\mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0)), \mathcal{O}(-h) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots \\ \dots \mathcal{O}(2h + H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle.$$

On the other hand with  $\tilde{\mathcal{H}}$  being the universal hyperplane section of  $\mathrm{Bl}_{\mathbb{P}(W)} \mathbb{P}^5$  with respect to  $\pi^* \mathcal{O}_{\mathbb{P}^5}(3)(-E)$ , it has a semiorthogonal decomposition of the form

$$\mathbf{D}^b(\tilde{\mathcal{H}}) = \langle \tilde{\mathcal{C}}, \tilde{\mathcal{A}}(1) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle$$

Replace  $\mathcal{A}$  with the exceptional collection  $\langle \mathcal{O}_{\mathbb{P}^5}(3), \mathcal{O}_{\mathbb{P}^5}(4), \mathcal{O}_{\mathbb{P}^5}(5) \rangle$  and  $\mathbf{D}^b(\mathbb{P}(W))$  with the exceptional collection  $\langle \mathcal{O}_{\mathbb{P}(W)}(-3), \mathcal{O}_{\mathbb{P}(W)}(-2), \mathcal{O}_{\mathbb{P}(W)}(-1) \rangle$ . The reason for these choices will be clear in a moment; writing this down explicitly we get:

$$\tilde{\mathcal{A}} = \langle \pi^* \mathcal{O}_{\mathbb{P}^5}(-3)(2E), \pi^* \mathcal{O}_{\mathbb{P}^5}(-2)(2E), \pi^* \mathcal{O}_{\mathbb{P}^5}(-1)(2E), j_* p^* \mathcal{O}_{\mathbb{P}^2}(-3)(2E), \\ j_* p^* \mathcal{O}_{\mathbb{P}^2}(-2)(2E), j_* p^* \mathcal{O}_{\mathbb{P}^2}(-1)(2E) \rangle.$$



We then get the following semiorthogonal decomposition for  $\mathbf{D}^b(\tilde{\mathcal{H}})$ :

$$(18) \quad \mathbf{D}^b(\tilde{\mathcal{H}}) = \langle \tilde{\mathcal{C}}', \pi^* \mathcal{O}_{\mathbb{P}^5} \boxtimes \mathbf{D}^b(P(L), \pi^* \mathcal{O}_{\mathbb{P}^5}(1) \boxtimes \mathbf{D}^b(\mathbb{P}(L))), \dots \\ \dots j_* \mathcal{O}_E \otimes \pi^* \mathcal{O}_{\mathbb{P}^5}(2) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle.$$

where we denote by  $\tilde{\mathcal{C}}'$  the category  $\tilde{\mathcal{C}} \otimes \mathcal{O}(-E, 0)$ . Note that in (18), to get the last three pieces of the semiorthogonal decomposition we used that

$$j_*(p^* \mathcal{O}_{\mathbb{P}(W)}(t) \otimes \mathcal{O}_E) \cong j_*(p^* i^* \mathcal{O}_{\mathbb{P}^5}(t) \otimes \mathcal{O}_E) \cong j_*(j^* \pi^* \mathcal{O}_{\mathbb{P}^5}(t) \otimes \mathcal{O}_E)$$

and the projection formula. Summing up, we get

$$(19) \quad \mathbf{D}^b(\tilde{\mathcal{H}}) = \langle \tilde{\mathcal{C}}', \mathcal{O} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \mathcal{O}(H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \\ \mathcal{O}(2H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), j_* \mathcal{O}_E \boxtimes \mathbf{D}^b(\mathbb{P}(L)), j_* \mathcal{O}_E(H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \\ j_* \mathcal{O}_E(2H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle$$

We are now ready to follow Kuznetsov's recipe for mutation [Kuz10, Section 4] to show that the interesting part  $\tilde{\mathcal{C}}'$  is equivalent to  $\mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0)$ .

*Step 1.* We start from (17) right mutating  $\Phi'(\mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0))$  through  $\mathcal{O}(-h) \boxtimes \mathbf{D}^b(\mathbb{P}(L))$ . This leads to the following decomposition:

$$(20) \quad \mathbf{D}^b(\tilde{\mathcal{H}}) = \langle \mathcal{O}(-h) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \Phi''(\mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0)), \\ \mathcal{O} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots \mathcal{O}(2h + H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle,$$

where  $\Phi'' = \mathbb{R}_{\mathcal{O}(-h) \boxtimes \mathbf{D}^b(\mathbb{P}(L))} \circ \Phi'$ .

*Step 2.* We now want to mutate  $\mathcal{O}(-h) \boxtimes \mathbf{D}^b(\mathbb{P}(L))$  through the left orthogonal subcategory  ${}^\perp \langle \mathcal{O}(-h) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle$ . The anticanonical bundle of  $\tilde{\mathcal{H}}$  is

$$\omega_{\tilde{\mathcal{H}}}^\vee = i_{\tilde{\mathcal{H}}}^* \left( \mathcal{O}(2H + h) \boxtimes \omega_{\mathbb{P}(L)}^\vee(-1) \right),$$

and thus we obtain

$$(21) \quad \mathbf{D}^b(\tilde{\mathcal{H}}) = \langle \Phi''(\mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0)), \\ \mathcal{O} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots \mathcal{O}(2h + H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \mathcal{O}(2H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle.$$

*Step 3.* Transpose the pair  $(\mathcal{O}(2h + H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \mathcal{O}(2H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)))$ . Using the Künneth formula we reduce to showing that

$$\mathrm{Ext}_{\tilde{\mathbb{P}^5}}^\bullet(\mathcal{O}(2h + H), \mathcal{O}(2H)) \boxtimes \mathrm{Ext}_{\mathbb{P}(L)}^\bullet(F, G)$$

vanishes for every  $F, G \in \mathbf{D}^b(\mathbb{P}(L))$ . Indeed we have

$$\begin{aligned} \mathrm{Ext}^\bullet(\mathcal{O}(2h + H), \mathcal{O}(2H)) &= H^\bullet(\mathrm{Bl}_{\mathbb{P}(W)} \mathbb{P}^5, \pi^* \mathcal{O}_{\mathbb{P}^5}(-1)(2E)) \\ &= H^\bullet(\mathbb{P}_{\mathbb{P}(W')}(\underline{W}(1) \oplus \mathcal{O}_{\mathbb{P}(W')}), \pi^* \mathcal{O}_{\mathbb{P}(W)}(-1) \otimes \mathcal{O}_\phi(1)) \\ &= H^\bullet(\mathbb{P}(W'), \mathcal{O}_{\mathbb{P}(W')}(-1) \otimes (\underline{W}(-1) \oplus \mathcal{O}_{\mathbb{P}(W')})) \\ &= 0. \end{aligned}$$

After transposing we have

$$(22) \quad \mathbf{D}^b(\tilde{\mathcal{H}}) = \langle \Phi''(\mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0)), \\ \mathcal{O} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots \mathcal{O}(2H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \mathcal{O}(2h + H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle.$$

*Step 4.* We left mutate  $\mathcal{O}(2h+H) \boxtimes \mathbf{D}^b(\mathbb{P}(L))$  through the right orthogonal subcategory  $\langle \mathcal{O}(2h+H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle^\perp$ . This just means twisting by the canonical bundle which we computed in Step 2:

$$(23) \quad \mathbf{D}^b(\tilde{\mathcal{H}}) = \langle \mathcal{O}(h-H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \Phi''(\mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0)), \mathcal{O} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots, \mathcal{O}(2H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle.$$

*Step 5.* Left mutation of  $\Phi''(\mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0))$  through  $\mathcal{O}(h-H) \boxtimes \mathbf{D}^b(\mathbb{P}(L))$  gives us the following decomposition:

$$(24) \quad \mathbf{D}^b(\tilde{\mathcal{H}}) = \langle \Phi'''(\mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0)), \mathcal{O}(h-H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \mathcal{O} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \dots, \mathcal{O}(2H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle,$$

where  $\Phi''' = \mathbb{L}_{\mathcal{O}(h-H) \boxtimes \mathbf{D}^b(\mathbb{P}(L))} \circ \Phi''$ . Let's remark at this point that the embedding  $\Phi'''$  is a kernel functor whose kernel is supported on  $(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0) \times_{\mathbb{P}(L)} \tilde{\mathcal{H}}$ . This can be seen as in the first example, except now one has to pay attention to the fact that we are dealing with a noncommutative variety. Using technical results of Kuznetsov [Kuz06b, Appendix D] one can check that everything goes through as expected.

*Step 6.* We want to simultaneously mutate  $\mathcal{O}(h-H) \boxtimes \mathbf{D}^b(\mathbb{P}(L))$  through  $\mathcal{O}_{\tilde{\mathbb{P}}^5} \boxtimes \mathbf{D}^b(\mathbb{P}(L))$ ,  $\mathcal{O}(h) \boxtimes \mathbf{D}^b(\mathbb{P}(L))$  through  $\mathcal{O}(H) \boxtimes \mathbf{D}^b(\mathbb{P}(L))$  and  $\mathcal{O}(h+H) \boxtimes \mathbf{D}^b(\mathbb{P}(L))$  through  $\mathcal{O}(2H) \boxtimes \mathbf{D}^b(\mathbb{P}(L))$ . As we already saw in the previous example, using Lemma 4.2 these mutations can be immediately computed. Write

$$\mathbf{D}^b(\mathbb{P}(L)) = \langle \mathcal{O}_{\mathbb{P}(L)}, \mathcal{O}_{\mathbb{P}(L)}(1), \dots, \mathcal{O}_{\mathbb{P}(L)}(l-1) \rangle$$

and perform the mutations objectwise. Using the Künneth formula we get for any  $0 \leq k' \leq k \leq l$ :

$$\dim \operatorname{Ext}^p(\mathcal{O}_{\tilde{\mathbb{P}}^5}(-E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k), \mathcal{O}_{\tilde{\mathbb{P}}^5} \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k')) = \begin{cases} 1 & \text{if } p = 0 \text{ and } k = k', \\ 0 & \text{else.} \end{cases}$$

The computation is the same as in Step 3. The first implication of this is that we can just swap pieces until we are in the situation of  $k = k'$ . In that case then we have the distinguished triangle

$$\mathbb{R}_{\mathcal{O}_{\tilde{\mathbb{P}}^5} \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)}(\mathcal{O}_{\tilde{\mathbb{P}}^5}(-E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)) \rightarrow \mathcal{O}_{\tilde{\mathbb{P}}^5}(-E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k) \rightarrow \mathcal{O}_{\tilde{\mathbb{P}}^5}$$

where the second map is the obvious one. It then follows that

$$\mathbb{R}_{\mathcal{O}_{\tilde{\mathbb{P}}^5} \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)}(\mathcal{O}_{\tilde{\mathbb{P}}^5}(-E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)) = j_* \mathcal{O}_E[-1] \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k),$$

and similarly one also shows that

$$\begin{aligned} \mathbb{R}_{\mathcal{O}_{\tilde{\mathbb{P}}^5}(H) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)}(\mathcal{O}_{\tilde{\mathbb{P}}^5}(h) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)) &= j_* \mathcal{O}_E(H)[-1] \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k), \\ \mathbb{R}_{\mathcal{O}_{\tilde{\mathbb{P}}^5}(2h) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)}(\mathcal{O}_{\tilde{\mathbb{P}}^5}(h+H) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)) &= j_* \mathcal{O}_E(2H)[-1] \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k). \end{aligned}$$

Finally, for any  $k' > k$  we see that  $\mathbb{R}_{\mathcal{O}_{\tilde{\mathbb{P}}^5} \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k)}(\mathcal{O}_{\tilde{\mathbb{P}}^5}(-E) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(k))$  and  $\mathcal{O}_{\mathbb{P}(L)}(k')$  are completely orthogonal and so we get that

$$\mathbb{R}_{\mathcal{O}_{\tilde{\mathbb{P}}^5} \boxtimes \mathbf{D}^b(\mathbb{P}(L))}(\mathcal{O}(h-H) \boxtimes \mathbf{D}^b(\mathbb{P}(L))) = \mathbb{R}_{\mathcal{O}_{\tilde{\mathbb{P}}^5}}(\mathcal{O}_{\tilde{\mathbb{P}}^5}(-E)) \boxtimes \mathbf{D}^b(\mathbb{P}(L)),$$

and similarly for the other two mutations. After this step we have the following semiorthogonal decomposition:

$$(25) \quad \mathbf{D}^b(\tilde{\mathcal{H}}) = \langle \Phi'''(\mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0)), \mathcal{O} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), j_* \mathcal{O}_E \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \mathcal{O}(H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), j_* \mathcal{O}_E(H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \mathcal{O}(2H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), j_* \mathcal{O}_E(2H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle.$$

*Step 7.* Finally we want to transpose  $j_*\mathcal{O}_E(H) \boxtimes \mathbf{D}^b(\mathbb{P}(L))$  to the right of  $\mathcal{O}(2H) \boxtimes \mathbf{D}^b(\mathbb{P}(L))$  and  $j_*\mathcal{O}_E \boxtimes \mathbf{D}^b(\mathbb{P}(L))$  to the right of  $\langle \mathcal{O}(H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \mathcal{O}(2H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle$ . We have to show complete orthogonality; this follows easily using the same argument of Kuznetsov [Kuz10, Lemma 4.6]. Indeed

$$(26) \quad \begin{aligned} \mathrm{Ext}_{\tilde{\mathcal{H}}}^\bullet(j_*\tilde{p}^*F, \tilde{\pi}^*G) &= \mathrm{Ext}_{\tilde{\mathcal{H}}}^\bullet(\tilde{p}^*F, \tilde{j}^*\tilde{\pi}^*G \otimes \mathcal{O}_{\tilde{E}}(-1)[-1]) = \\ &= \mathrm{Ext}_{\tilde{\mathcal{H}}}^\bullet(\tilde{p}^*F, \tilde{p}^*\tilde{i}^*G \otimes \mathcal{O}_{\tilde{E}}(-1)[-1]) = \mathrm{Ext}_{\tilde{\mathcal{H}}}^\bullet(F, \tilde{i}^*G \otimes \tilde{p}_*\mathcal{O}_{\tilde{E}}(-1)[-1]) = 0 \end{aligned}$$

where we used the notation of diagram (5) and the adjunction formula for  $j_*$ . So finally we get:

$$(27) \quad \begin{aligned} \mathbf{D}^b(\tilde{\mathcal{H}}) &= \langle \Phi'''(\mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0), \mathcal{O} \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \mathcal{O}(H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), \\ &\quad \mathcal{O}(2H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)), j_*\mathcal{O}_E \boxtimes \mathbf{D}^b(\mathbb{P}(L)), j_*\mathcal{O}_E(H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \\ &\quad j_*\mathcal{O}_E(2H) \boxtimes \mathbf{D}^b(\mathbb{P}(L)) \rangle \end{aligned}$$

Comparing with (19), we conclude that  $\tilde{\mathcal{C}}' \cong \Phi'''(\mathbf{D}^b(\mathbb{P}(W') \times \mathbb{P}(L), \mathcal{C}_0))$ .

*Remark 4.7.* Note again by Theorem 3.8 that this is just a family version of Kuznetsov's result on cubic fourfolds containing a plane [Kuz10]. Again by taking a generic pencil we also recover the corresponding result of Calabrese and Thomas [CT15].

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